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STRICTLY CONJUGATE STRESS AND STRAIN TENSORS

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A subclass of strictly conjugate tensors, namely, the tensors that satisfy the requirement for transformation by the same law upon rigid motion of the neighborhood of a material particle, is separated into the class of work-conjugate stress and strain tensors. The advantage of the use of strictly conjugate stress and strain tensors in formulating the variational principles for bodies from a hyperelastic material is shown.

Introduction. At present, various stress and strain tensors are used in continuum mechanics. The first attempts to put a certain stress tensor into correspondence with a certain strain tensor with the use of the work of internal forces were made in [1]. The stress and strain tensors that are work-conjugate were defined in [2–7]. In the present work, we propose to introduce a subclass of work-conjugate stress and strain tensors that meet the requirement for transformation by the same law of conjugate stress and strain tensors upon rigid motions of the neighborhood of a material particle. It is shown that when this restriction is not satisfied, some known functionals for the variational equations of the nonlinear theory of elasticity do not possess the property of invariance relative to the rigid motions entering the integrands of the terms.

1. Kinematics of Deformation. Let $\boldsymbol{X}$ and $\boldsymbol{x}$ be the position vectors of a material particle of a deformable body in the reference and current configurations, respectively, and $\boldsymbol{u} \equiv \boldsymbol{x} - \boldsymbol{X}$ be the displacement vector of this particle. We introduce the asymmetric deformation-gradient F and displacement-gradient H tensors and the corresponding transposed tensors $\bar{F}$ and $\bar{H}$ [3, 5–8]:

$$F \equiv \boldsymbol{x}\nabla, \qquad H \equiv \boldsymbol{u}\nabla, \qquad \bar{F} \equiv \nabla\boldsymbol{x}, \qquad \bar{H} \equiv \nabla\boldsymbol{u}. \tag{1.1}$$

These tensors are related by the relations

$$F = g + H, \qquad \bar{F} = g + \bar{H}, \qquad \bar{F} = F^{\mathrm{t}}, \qquad \bar{H} = H^{\mathrm{t}}. \tag{1.2}$$

Here g is the metric (unit) tensor, ∇ is the Hamiltonian operator in the metric of the reference configuration of a body, and the superscript "t" refers to transposition. In the Cartesian coordinate system with the orthonormalized basis vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, and $\boldsymbol{k}_3$, the Hamiltonian operator is defined as $\nabla \equiv \boldsymbol{k}_i \partial/\partial X_i$ and the tensors F, $\bar{F}$, H, and $\bar{H}$ have the following representations:

$$F = x_{i,j}\boldsymbol{k}_i\boldsymbol{k}_j, \qquad \bar{F} = x_{j,i}\boldsymbol{k}_i\boldsymbol{k}_j, \qquad H = u_{i,j}\boldsymbol{k}_i\boldsymbol{k}_j, \qquad \bar{H} = u_{j,i}\boldsymbol{k}_i\boldsymbol{k}_j,$$

where $(\cdot)_{,i} \equiv \partial(\cdot)/\partial X_i$, X_i, x_i, and u_i $(i = 1, 2, 3)$ are the components of the vectors $\boldsymbol{X}$, $\boldsymbol{x}$, and $\boldsymbol{u}$, respectively; hereinafter, summation is performed over the repeated indices.

The asymmetric tensors F, $\bar{F}$, H, and $\bar{H}$ characterize the deformation of the neighborhood of a material particle that includes its distortion and rotation. We consider the symmetric Green–Lagrange strain tensor which determines distortion of the neighborhood of the material particle [5–8]:

$$E \equiv \frac{1}{2}(\bar{F}\cdot\bar{F}^{\mathrm{t}} - g) = \frac{1}{2}(\bar{H} + \bar{H}^{\mathrm{t}} + \bar{H}\cdot\bar{H}^{\mathrm{t}}) = \frac{1}{2}(F^{\mathrm{t}}\cdot F - g) = \frac{1}{2}(H + H^{\mathrm{t}} + H^{\mathrm{t}}\cdot H). \tag{1.3}$$

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Hereinafter, the point between the tensors denotes the scalar (internal) product of the tensors.

2. Stress Tensors. The symmetric true Cauchy stress tensor is denoted by s. We introduce the nominal-stress tensors expressed in terms of the tensor s and the kinematic tensors introduced above [2–9]:

$$\tau \equiv Js, \qquad S \equiv F^{-1} \cdot \tau \cdot \bar{F}^{-1}, \qquad P \equiv \tau \cdot \bar{F}^{-1}, \qquad \bar{P} \equiv F^{-1} \cdot \tau = P^{\mathrm{t}},$$

where $J \equiv \rho/\bar{\rho} = \det F$, ρ and $\bar{\rho}$ are the mass densities of a material in the reference and current configurations, τ is the symmetric Kirchhoff stress tensor, S is the symmetric second Piola–Kirchhoff stress tensor, P is the asymmetric first Piola–Kirchhoff stress tensor, and $\bar{P}$ is the asymmetric Lagrange stress tensor.

3. Conjugate Stress and Strain Tensors. The work of the internal forces per unit mass of a body is determined in the form [1–7, 9]

$$w \equiv \frac{1}{\bar{\rho}} s : d = \frac{1}{\rho} \tau : d, \tag{3.1}$$

where $d \equiv (l + l^{\mathrm{t}})/2$ is the symmetric strain-rate tensor, $l = \dot{F} \cdot F^{-1}$ is the velocity-gradient tensor [6, 10], the dot denotes the material derivative, ":" denotes the double scalar product of arbitrary tensors of the second rank A and B: $A : B \equiv \operatorname{tr}(A \cdot B^{\mathrm{t}})$, and $\operatorname{tr} h$ is the first invariant of the tensor of the second rank h.

Definition 1. We call the stress tensor A and the strain tensor B *work-conjugate tensors* if the equality

$$A : \dot{B} = \tau : d$$

holds. The equalities [2–9]

$$\tau : d = S : \dot{E} = \bar{P} : \dot{\bar{F}} = P : \dot{F} = \bar{P} : \dot{\bar{H}} = P : \dot{H} \tag{3.2}$$

are true, i.e., the pairs of stress and strain tensors

$$(S, E), \qquad (\bar{P}, \bar{F}), \qquad (P, F), \qquad (\bar{P}, \bar{H}), \qquad (P, H) \tag{3.3}$$

are conjugate. One can introduce other pairs of conjugate stress and strain tensors [1–7].

We consider two body motions determined by the laws $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ and $\boldsymbol{x}^*(\boldsymbol{X}, t)$, where t is a monotonically increased deformation parameter. If there is a neighborhood in which the equality

$$\boldsymbol{x}^*(\boldsymbol{X}, t) = Q(t) \cdot \boldsymbol{x}(\boldsymbol{X}, t) + \boldsymbol{c}(t)$$

holds for a certain material particle, the motion of this neighborhood from $\boldsymbol{x}$ to $\boldsymbol{x}^*$ (or vice versa) is called rigid motion [5, 6, 8, 10]. Here $Q(t)$ is the proper orthogonal tensor ($Q^{\mathrm{t}} = Q^{-1}$, $\det Q = 1$), which corresponds to the rotation of this neighborhood, and the vector $\boldsymbol{c}(t)$ corresponds to its displacement.

Definition 2. The stress and strain tensors are called *strictly conjugate* tensors if the work-conjugate stress and strain tensors are transformed under the same law upon rigid motions of the neighborhood of a material particle.

Upon rigid motions of the neighborhood of a material particle, the stress and strain tensors introduced above are transformed by the following formulas [5, 6, 8, 10, 11]:

$$\begin{aligned} &E^* = E, \qquad \bar{F}^* = \bar{F} \cdot Q^{\mathrm{t}}, \qquad F^* = Q \cdot F, \qquad d^* = Q \cdot d \cdot Q^{\mathrm{t}}, \\ &S^* = S, \qquad \bar{P}^* = \bar{P} \cdot Q^{\mathrm{t}}, \qquad P^* = Q \cdot P, \qquad \tau^* = Q \cdot \tau \cdot Q^{\mathrm{t}}. \end{aligned} \tag{3.4}$$

We introduce the laws of transformation of the tensors $\bar{H}$ and H upon rigid motions of the neighborhood of a material particle. From (1.2) and (3.4), we obtain the relations

$$\bar{F}^* = g^* + \bar{H}^*, \qquad \bar{F}^* = (g + \bar{H}) \cdot Q^{\mathrm{t}}. \tag{3.5}$$

Taking into account the equality $g^* = g$, from (3.5) we obtain

$$\bar{H}^* = Q^{\mathrm{t}} - g + \bar{H} \cdot Q^{\mathrm{t}}, \qquad H^* = \bar{H}^{*\mathrm{t}} = Q - g + Q \cdot H. \tag{3.6}$$

It follows from (3.4) and (3.6) that of the conjugate pairs of tensors in (3.3) the pairs (S, E), $(\bar{P}, \bar{F})$, and (P, F) are strictly conjugate.

4. Constitutive Relations of a Hyperelastic Material. In the classification of constitutive relations for finite (large) strains, three types of elastic materials are distinguished [9, 12]: hyperelastic, elastic, and hypoelastic. Under infinitesimal deformations of a body from a linear elastic material, the three formulations are equivalent [9]. According to [9, 12], we give the definition of a hyperelastic material.

Definition 3. The material of a body is called a hyperelastic material if there exists a natural configuration of the body in which the stresses and the strains are equal to zero and an analytical function $W(E)$ formed relative to the natural configuration such that the equality

$$\dot{W} = w. \tag{4.1}$$

holds at any point of the body. The mechanical meaning of the function W follows from the laws of thermodynamics: this function is a strain energy function per unit mass of a body. It is convenient to introduce the strain energy density $\bar{W}(E)$ [9] (strain energy per unit reference body volume):

$$\bar{W} \equiv \rho W. \tag{4.2}$$

From (4.1), (4.2), (3.1), and (3.2) follow the constitutive relations of a hyperelastic material:

$$S = \frac{d\bar{W}}{dE}. \tag{4.3}$$

The alternate forms of writing the constitutive relations of a hyperelastic material can be obtained with the use of other conjugate pairs of stress and strain tensors. Using (1.3), we derive the expressions of the strain-energy density:

$$\bar{E}(\bar{F}) \equiv \bar{W}[E(\bar{F})], \quad \tilde{E}(F) \equiv \bar{W}[E(F)], \quad \hat{E}(\bar{H}) \equiv \bar{W}[E(\bar{H})], \quad \check{E}(H) \equiv \bar{W}[E(H)]. \tag{4.4}$$

From (4.1), (4.2), (3.1), (3.2). and (4.4) follow the representations of the constitutive relations of a hyperelastic material [5, 7, 9, 12–15]:

$$\bar{P} = \frac{d\bar{E}(\bar{F})}{d\bar{F}}, \qquad P = \frac{d\tilde{E}(F)}{dF}, \qquad \bar{P} = \frac{d\hat{E}(\bar{H})}{d\bar{H}}, \qquad P = \frac{d\check{E}(H)}{dH}. \tag{4.5}$$

5. Equations of Motion. The equations of motion with boundary conditions are written by means of the stress tensor $\bar{P}$ [5, 6, 9, 12–15]

$$\nabla \cdot \bar{P} + \rho \boldsymbol{f} = \rho \ddot{\boldsymbol{u}} \text{ in } V, \qquad \boldsymbol{u} = \boldsymbol{u}^p \text{ on } S_u, \qquad \boldsymbol{N} \cdot \bar{P} = \boldsymbol{T}^p \text{ on } S_T \tag{5.1}$$

or P [8]:

$$P \cdot \nabla + \rho \boldsymbol{f} = \rho \ddot{\boldsymbol{u}} \text{ in } V, \qquad \boldsymbol{u} = \boldsymbol{u}^p \text{ on } S_u, \qquad P \cdot \boldsymbol{N} = \boldsymbol{T}^p \text{ on } S_T. \tag{5.2}$$

Here V is the reference configuration of the body, S_u and S_T are parts of the surface S ($S = S_u \cup S_T$) that bound the region V, on which the components of the displacement vector $\boldsymbol{u}$ and the stress vector $\boldsymbol{T} \equiv \boldsymbol{N} \cdot \bar{P} = P \cdot \boldsymbol{N}$ are specified, $\boldsymbol{N}$ is the unit vector of the external normal to the surface S, and $\nabla \cdot \bar{P}$ and $P \cdot \nabla$ are the notation of the operations of divergence of the tensors $\bar{P}$ and P, respectively (in the Cartesian coordinate system, we have $\nabla \cdot \bar{P} = \bar{P}_{ji,j}\boldsymbol{k}_i$ and $P \cdot \nabla = P_{ij,j}\boldsymbol{k}_i$), and $\boldsymbol{f}$ is the mass-force vector; the superscript p denotes a prescribed quantity.

The equations of motion with the use of the stress tensor S are derived after the expressions

$$\bar{P} = S \cdot \bar{F} = S \cdot (g + \nabla \boldsymbol{u}) \tag{5.3}$$

are substituted into (5.1) or the expressions

$$P = F \cdot S = (g + \boldsymbol{u}\nabla) \cdot S$$

are substituted into system (5.2).

6. Systems of Hyperelasticity Equations. Various forms of writing the closed systems of equations that describe deformable bodies from hyperelastic materials are obtained with the use of an appropriate conjugate pair of stress and strain tensors. Table 1 lists equations that form these closed systems for the pairs of conjugate tensors considered.

TABLE 1

Pair	Equations
S, E	(5.1), (5.3), (4.3), (1.3), (1.1)
$\bar{P}, \bar{F}$	(5.1), (4.5), (1.2), (1.1)
P, F	(5.2), (4.5), (1.2), (1.1)
$\bar{P}, \bar{H}$	(5.1), (4.5), (1.1)
P, H	(5.2), (4.5), (1.1)

7. Complementary Strain Energy Density. We assume that one can invert the constitutive relations (4.3) and (4.5):

$$E = E(S), \quad \bar{F} = \bar{F}(\bar{P}), \quad F = F(P), \quad \bar{H} = \bar{H}(\bar{P}), \quad H = H(P).$$

Then, using the Legendre transformation, one can introduce functions, namely, the *complementary strain energy densities*:

$$\bar{W}_c = S : E - \bar{W}, \quad \bar{E}_c = \bar{P} : \bar{F} - \bar{E}, \quad \tilde{E}_c = P : F - \tilde{E}, \quad \hat{E}_c = \bar{P} : \bar{H} - \hat{E}, \quad \check{E}_c = P : H - \check{E}, \tag{7.1}$$

so that the inverted constitutive relations can be written in the form [5, 9, 14]

$$E = \frac{d\bar{W}_c(S)}{dS}, \quad \bar{F} = \frac{d\bar{E}_c(\bar{P})}{d\bar{P}}, \quad F = \frac{d\tilde{E}_c(P)}{dP}, \quad \bar{H} = \frac{d\hat{E}_c(\bar{P})}{d\bar{P}}, \quad H = \frac{d\check{E}_c(P)}{dP}. \tag{7.2}$$

It follows from (7.1) that

$$\begin{gathered} \bar{W} = \underline{S : E} - \bar{W}_c, \qquad \bar{E} = \underline{\bar{P} : \bar{F}} - \bar{E}_c, \qquad \tilde{E} = \underline{P : F} - \tilde{E}_c, \\ \hat{E} = \bar{P} : \bar{H} - \hat{E}_c, \qquad \check{E} = P : H - \check{E}_c. \end{gathered} \tag{7.3}$$

Using (3.4), for rigid motions we obtain

$$S^* : E^* = S : E, \qquad \bar{P}^* : \bar{F}^* = \bar{P} : \bar{F}, \qquad P^* : F^* = P : F.$$

Thus, for strictly conjugate pairs of stress and strain tensors, the underlined terms on the right sides of (7.3) are invariant (unvaried upon rigid motions) quantities. The strain energy densities on the left sides of (7.3) are invariant quantities. Therefore, the complementary strain energy densities $\bar{W}_c$, $\bar{E}_c$, and $\tilde{E}_c$ are invariant quantities. Generally, the first terms on the right sides of the fourth and fifth formulas in (7.3) are not invariant quantities; therefore, the complementary strain energy densities $\hat{E}_c$ and $\check{E}_c$ are noninvariant quantities as well.

8. Variational Principles of the Nonlinear Theory of Elasticity. We analyze different variants of writing of the functionals of two known variational principles of the nonlinear theory of elasticity (under the assumption that $\ddot{\boldsymbol{u}} = \boldsymbol{0}$ for static problems), namely, the principle of stationarity of the potential energy and the principle of stationarity of the complementary energy. In the sequel, it is assumed that the external-force vectors $\boldsymbol{f}$ and $\boldsymbol{T}^p$ do not depend on the displacement vector $\boldsymbol{u}$.

For all the pairs of conjugate tensors considered, the functional of potential energy of a body from a hyperelastic material in the class of quite smooth displacement fields $\boldsymbol{u}$ which satisfy the kinematic boundary conditions in (5.1) and (5.2), has the same form:

$$I(\boldsymbol{u}) \equiv \int\limits_V [E(\boldsymbol{u}) - \rho \boldsymbol{f} \cdot \boldsymbol{u}]\, dV - \int\limits_{S_T} \boldsymbol{T}^p \cdot \boldsymbol{u}\, dS.$$

Here $E(\boldsymbol{u}) \equiv \bar{W}[E(\nabla \boldsymbol{u})] = \bar{E}[\bar{F}(\nabla \boldsymbol{u})] = \tilde{E}[F(\boldsymbol{u}\nabla)] = \hat{E}(\nabla \boldsymbol{u}) = \check{E}(\boldsymbol{u}\nabla)$.

The variational equation of the principle of stationarity of the potential energy is written in the form [5, 9, 14]

$$\delta I(\boldsymbol{u}) = 0. \tag{8.1}$$

The equations of the Euler and natural boundary conditions of the variational equation (8.1) are the equations of equilibrium and the static boundary conditions in (5.1) or (5.2) with allowance for the dependences $\bar{P} = \bar{P}(\nabla \boldsymbol{u})$ or $P = P(\boldsymbol{u}\nabla)$, which are obtained when the constitutive relations of a hyperelastic material in (4.5) and the kinematic constrains in (1.1) and (1.2) hold.

Consideration of the variational principle of stationarity of the complementary energy only for asymmetric conjugate pairs of stress and strain tensors make sense [14]. We consider the functional (complementary energy) proposed in [16] in the class of sufficiently smooth fields of the Lagrange stress tensor $\bar{P}$, which satisfy the equations of equilibrium ($\ddot{\boldsymbol{u}} = \boldsymbol{0}$) and the static boundary conditions in (5.1):

$$\bar{J}(\bar{P}) \equiv \int\limits_V \bar{E}_c(\bar{P})\, dV - \int\limits_{S_u} \boldsymbol{N} \cdot \bar{P} \cdot \boldsymbol{x}\, dS. \tag{8.2}$$

The variational equation of the complementary-energy principle is written in the form [16]

$$\delta \bar{J}(\bar{P}) = 0. \tag{8.3}$$

The Euler equation of the variational equation (8.3) is a compatibility equation, which is the second equation in (7.2), where $\bar{F} \equiv \nabla \boldsymbol{x}$.

If one uses the corresponding fields of the first Piola–Kirchhoff stress tensor, which satisfy the equations of equilibrium and the static boundary conditions in (5.2), instead of the varied fields of the Lagrange stress tensor $\bar{P}$, P, the functional (8.2) should be replaced by the functional

$$\tilde{J}(P) \equiv \int\limits_V \tilde{E}_c(P)\, dV - \int\limits_{S_u} \boldsymbol{x} \cdot P \cdot \boldsymbol{N}\, dS. \tag{8.4}$$

The third equation in (7.2), where $\delta \tilde{J}(P) = 0$, is the Euler equation of the variational equation $F \equiv \boldsymbol{x}\nabla$.

We consider the functional [14]

$$\hat{J}(\bar{P}) \equiv \int\limits_V \hat{E}_c(\bar{P})\, dV - \int\limits_{S_u} \boldsymbol{N} \cdot \bar{P} \cdot \boldsymbol{u}\, dS. \tag{8.5}$$

The fourth equation in (7.2), where $\delta \hat{J}(\bar{P}) = 0$, is the Euler equation of the variational equation $\bar{H} \equiv \nabla \boldsymbol{u}$.

For the variational equation $\delta \check{J}(P) = 0$ with the functional

$$\check{J}(P) \equiv \int\limits_V \check{E}_c(P)\, dV - \int\limits_{S_u} \boldsymbol{u} \cdot P \cdot \boldsymbol{N}\, dS, \tag{8.6}$$

the fifth equation in (7.2), where $H \equiv \boldsymbol{u}\nabla$, is the Euler equation.

It is noteworthy that the integrands in the volumetric integral are invariant quantities in the functionals (8.2) and (8.4) and noninvariant quantities in the functionals (8.5) and (8.6).

Conclusions. In the present study, the notion of strictly conjugate stress and strains tensors has been introduced; the use of these tensors does not lead to noninvariant expressions for complementary strain energy density. The conjugate pair of tensors $(\bar{P}, \bar{H})$, which does not enter the class of strictly conjugate tensors, is used in some studies (cf. [9, 13–15]). The use of strictly conjugate pairs $(\bar{P}, \bar{F})$ [5, 6, 12, 16] or (P, F) [8, 11] is preferred in the consideration of hyperelastic materials.

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